Temporal logic control of nonlinear stochastic systems using a piecewise-affine abstraction

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***Abstract*— Automatically synthesizing controllers for continuous-state nonlinear stochastic systems, while giving guarantees on the probability of satisfying a temporal logic specification crucially depends on abstractions with a quantified accuracy. For this similarity quantification, approximate stochastic simulation relations are often used. To handle the nonlinearity of the system effectively, we use finite-state abstractions based on piecewise-affine approximations together with tailored simulation relations that leverage the local affine structure. We end this paper by synthesizing a robust controller for a nonlinear stochastic Van der Pol oscillator.**

I. Introduction

The design of controllers for safety-critical systems, such as airplanes, cars and power systems, requires guarantees on their correct functioning. Although obtaining guarantees on their behavior via analysis and verification is important, many of these systems are difficult to analyze and verify as they evolve over continuous spaces in a stochastic and generally nonlinear fashion. Therefore, we need methods that can handle simultaneously complex safety-critical requirements, large scale continuous states, and stochastic and nonlinear state evolutions. Recent work [1], [2], has shown progress in the design of methods based on temporal logic specifications that can handle relevant safety specifications. Although these approaches scale to, respectively, more complex specifications [1] and larger stochastic systems with continuous states [2], they are still limited to linear or pseudo-linear stochastic systems.

For the nonlinear stochastic difference equations considered in this work, less progress has been shown. Synthesizing a provably correct controller that guarantees the satisfaction of temporal logic specifications for nonlinear stochastic systems remains a very challenging problem and the amount of methods that exist is very limited. More specifically, methods either focus on a very specific type of specification [3]–[5] or use slope restrictions on the nonlinearity (pseudo-linearity) of the systems [2]. When focusing on local behavior, many nonlinear systems behave almost linear. Therefore, a widely adopted approach in classical control is performing a *piecewise-affine approximation* of the nonlinear system [6]–[8]. In this paper, we leverage piecewise-affine approximations to synthesize a controller, while simultaneously computing the satisfaction probability of a temporal logic specification.

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To apply such a formal synthesis method, which has guarantees, the continuous-state behavior of safety-critical systems is often approximated by a finite-state model [9], known as an abstraction. By quantifying the similarity between the original, continuous-state model and its finite-state abstraction, it is possible to obtain guarantees on the satisfaction of formal specifications. As in [1], [10], the similarity or deviations in probability and output of stochastic systems can be expressed using approximate simulation relations [11]. In this work, we will use this theory, applicable to both linear and nonlinear stochastic systems, and develop tailored methods for the provably correct controller design of nonlinear stochastic systems.

Literature. The existing methods for temporal logic verification and control of nonlinear stochastic systems can be classified into abstraction-based and abstraction-free methods. As mentioned before, available results on abstraction-based methods for nonlinear stochastic systems [2], [12], [13] are either restricted in the type of systems or with respect to the specification. More specifically, [2] is restricted to systems satisfying strict dissipativity requirements that are pseudo-linear, that is, systems whose nonlinearity has a bounded slope. Furthermore, this method tends to yield conservative results. The tool FAUST^2 [12] can only handle specifications with a finite horizon. Similarly, [13] can only handle liveness or repeated reachability specifications. On the other hand abstraction-free methods that directly synthesize controllers for the continuous-state systems are generally based on Barrier certificates [3]–[5] and are limited to safety specifications.

The work on deterministic piecewise-affine (PWA) approximations is well established [6]–[8], [15] and has applications in multiple scientific domains [16]–[22]. Early PWA approximation methods use a uniform partitioning of the continuous state space [6], [15], [23], while more recent methods consider the curvature or the variation of the nonlinear functions [24]–[26].

As a first step to incorporate piecewise-affine approximations into the temporal logic control of stochastic systems, we use the standard piecewise-affine approximation technique based on Taylor series [27, Sec. 4.10 & 9.6]. Based on this, we develop a method to synthesize a provably correct controller for nonlinear stochastic systems that allows us to apply computationally efficient methods similar to [28] to locally quantify the probability deviation, while maintaining a global output deviation. To achieve this, we construct

a local affine approximation of the nonlinear dynamics, quantify the error of this approximation and construct a piecewise-affine finite-state abstraction for the complete state space (Section III). In Section IV, we discuss how to go from local error dynamics to a global similarity quantification by defining a piecewise simulation relation. Next, we adjust the dynamic programming mapping from [29] to compute a robust satisfaction probability of the temporal logic specification (Section V). Finally, we design a controller for a stochastically perturbed Van der Pol Oscillator and discuss the results in Section V.

II. Problem formulation and approach

For a given set $\mathbb{X}$ in Euclidean space[1], the Borel measurable space is denoted as $(\mathbb{X}, \mathcal{B}(\mathbb{X})$ with $\mathcal{B}(\mathbb{X})$ the σ-algebra of the Borel sets [30]. A probability measure $\mathbb{P}$ over this space has realizations $x \sim \mathbb{P}$ with $x \in \mathbb{X}$. The set of probability measures on the measurable space $(\mathbb{X}, \mathcal{B}(\mathbb{X})$ is denoted by $\mathcal{P}(\mathbb{X})$. The weighted two-norm $||x||_D$ is defined as $||x||_D = \sqrt{x^\top D x}$. Furthermore, I_N denotes the identity matrix of size $\mathbb{R}^N \times \mathbb{R}^N$. The Minkowski sum of two sets A and B is defined as $A \oplus B := \{a + b \mid a \in A, b \in B\}$.

A. Preliminaries

Model. Consider a system whose behavior can be modeled by a discrete-time nonlinear stochastic difference equation

$$M : \begin{cases} x_{t+1} & = f(x_t) + Bu_t + w_t \\ y_t & = Cx_t, \quad \forall t \in \{0, 1, 2, \dots\}, \end{cases} \tag{1}$$

with state $x \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$, input $u \in \mathbb{U} \subseteq \mathbb{R}^{n_u}$ disturbance $w \in \mathbb{W} \subseteq \mathbb{R}^{n_w}$ and output $y \in \mathbb{Y} \subseteq \mathbb{R}^{n_y}$. Furthermore, we have matrices $B \in \mathbb{R}^{n_x \times n_u}, C \in \mathbb{R}^{n_y \times n_x}$ and the nonlinear function $f : \mathbb{X} \to \mathbb{X}$ is assumed to be measurable and sufficiently smooth. The disturbance w_t is an independently and identically distributed (i.i.d.) noise signal with realizations $w \sim \mathbb{P}_w$ and the system is initialized at $x_0 \in \mathbb{X}$. For simplicity, we have assumed that the output and input enter in a linear way. This assumption is made without loss of generality since systems with nonlinear terms $g(x_t)u_t$ and $h(x_t)$ instead of Bu_t and Cx_t respectively, can be handled in a similar fashion.

A (finite) path $\boldsymbol{\omega}_{\to t} := x_0, u_0, x_1, u_1, \dots x_t$ of a system is built up from inputs u_t and from realizations x_{t+1} based on (1) for a given state x_t, input u_t and disturbance w_t for each time step t. A control strategy $\boldsymbol{\mu} := \mu_0, \mu_1, \mu_2, \dots$ consists of maps $\mu_t(\boldsymbol{\omega}_{\to t})$ that determines an input u_t for each finite path of the model (1). In this work, we are focusing on control strategies $\boldsymbol{C}$ that can be represented with finite memory.

Specification. To express formal specifications, we use syntactically co-safe linear temporal logic (scLTL) [9], [31]. This language consists of atomic propositions $p_1, p_2, \dots, p_N$ that are either true or false. The set of atomic propositions is denoted as $AP = \{p_1, \dots, p_N\}$ and it defines an alphabet 2^{AP}. Together the set of atomic propositions that are true form a letter in the alphabet, that is, $\pi \in 2^{AP}$. A word $\boldsymbol{\pi} = \pi_0 \pi_1 \pi_2 \dots$ is formed by a (possibly infinite) string of letters with associated suffix $\boldsymbol{\pi}_t = \pi_t \pi_{t+1} \pi_{t+2} \dots$. It is over these words that specifications are checked. Such a formal specification, written as a temporal logic formula, is formed by combining atomic propositions with logical and temporal operators as defined in the scLTL syntax.

Definition 1 (scLTL syntax): *An scLTL formula ϕ is defined over a set of atomic propositions as*

$$\phi ::= p \mid \neg p \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \bigcirc \phi \mid \phi_1 \mathsf{U} \phi_2,$$

with atomic proposition $p \in AP$. □

The semantics of this syntax can be given for the suffices $\boldsymbol{\pi}_t$. An atomic proposition $\boldsymbol{\pi}_t \models p$ holds if $p \in \pi_t$, while a negation $\boldsymbol{\pi}_t \models \neg\phi$ holds if $\boldsymbol{\pi}_t \not\models \phi$. Furthermore, a conjunction $\boldsymbol{\pi}_t \models \phi_1 \wedge \phi_2$ holds if both $\boldsymbol{\pi}_t \models \phi_1$ and $\boldsymbol{\pi}_t \models \phi_2$ are true, while a disjunction $\boldsymbol{\pi}_t \models \phi_1 \vee \phi_2$ holds if either $\boldsymbol{\pi}_t \models \phi_1$ or $\boldsymbol{\pi}_t \models \phi_2$ is true. Also, a next statement $\boldsymbol{\pi}_t \models \bigcirc\phi$ holds if $\boldsymbol{\pi}_{t+1} \models \phi$. Finally, an until statement $\boldsymbol{\pi}_t \models \phi_1 \mathsf{U} \phi_2$ holds if there exists an $i \in \mathbb{N}$ such that $\boldsymbol{\pi}_{t+i} \models \phi_2$ and for all $j \in \mathbb{N}, 0 \leq j < i$ we have $\boldsymbol{\pi}_{t+j} \models \phi_1$. Via a labeling function $L : \mathbb{Y} \to 2^{AP}$, an output trajectory $\boldsymbol{y} = y_0 y_1 y_2 \dots$ of a system (1) is translated to a word $\boldsymbol{\pi} = L(y_0)L(y_1)L(y_2)\dots$. As such a system satisfies a specification if the generated word $\boldsymbol{\pi}_0 = \boldsymbol{\pi} = L(\boldsymbol{y})$ satisfies the specification, i.e., $\boldsymbol{\pi}_0 \models \phi$.

B. Problem statement

The goal of this work is to automatically develop a controller $\boldsymbol{C}$, such that the controlled system $M \times \boldsymbol{C}$ satisfies a specification ϕ. Since we are considering stochastic systems, we are interested in the satisfaction probability of a specification, which is the probability that words generated by the controlled system satisfy a specification, denoted as $\mathbb{P}(M \times \boldsymbol{C} \models \phi)$.

Problem. Given model M as in (1), an scLTL specification ϕ and a probability $p \in [0, 1]$, design a controller $\boldsymbol{C}$, such that

$$\mathbb{P}(M \times \boldsymbol{C} \models \phi) \geq p. \tag{2}$$

We approach this problem by gridding the continuous-state space after locally approximating the nonlinearity of the model using a piecewise-affine function in Section III. This yields a global finite-state abstraction of the original nonlinear model that is piecewise-affine. To compare the nonlinear model and the piecewise-affine abstract model, we locally couple the two models and define a piecewise approximate stochastic simulation relation similar to [11], whose computation is implicitly based on invariant set computations based on [28].

III. Global abstraction as a piecewise-affine system

In this section, we discuss the first step in designing a provably correct controller, namely constructing a piecewise-

[1]In this work, we limit our results to sets in Euclidean spaces, which are measurable and separable spaces.

affine abstraction of the nonlinear system in (1).

Local affine approximation of $f(x_t)$. In order to handle the non-linearity of $f(x_t)$ in (1), we can use affine functions to locally approximate it in the bounded set $\mathbb{G}$. Taylor's Theorem [27, Sec. 4.10] states that a function $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ that is infinitely differentiable at the point ν can be written as a power series. For a one-dimensional (1D) function $f : \mathbb{R} \to \mathbb{R}$, this power series equals

$$\begin{aligned} f(x) &= f(\nu) + \frac{f'(\nu)}{1!}(x-\nu) + \ldots + \frac{f^{N_T}(\nu)}{N_T!}(x-\nu)^{N_T} + \ldots \\ &= \sum_{l=0}^{\infty} \frac{f^{(l)}(\nu)}{l!}(x-\nu)^l. \end{aligned} \tag{3}$$

Here, $f^{(l)}(\nu)$ is the l-th derivative of $f(x)$ evaluated at $x = \nu$. A similar expression can be derived for multivariate function $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ from [32, Sec. 2.4], however, for brevity this expression is omitted.

Based on (3), the N_Tth-degree Taylor polynomial of a 1D function equals

$$f_{N_T}(x) = f(\nu) + \sum_{l=1}^{N_T} \frac{f^{(l)}(\nu)}{l!}(x-\nu)^l. \tag{4}$$

Here, N_T is a positive integer. From the multivariate version in [32], we can conclude that if $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is $N_T + 1$ times differentiable on the set $\mathbb{G} \subset \mathbb{R}^{n_x}$ that contains a vector $\nu \in \mathbb{R}^{n_x}$ in its interior, then the function $f(x_t)$ can locally be approximated by the N_Tth-order Taylor polynomial, that is, $f(x_t) \approx f_{N_T}(x_t)$ for $x_t \in \mathbb{G}$. In case $N_T = 1$, then the first-order Taylor polynomial is an affine function

$$f(x_t) \approx f_1(x_t) = Ax_t + a \text{ for } x \in \mathbb{G}, \tag{5}$$

with matrix $A = \nabla f(\nu)$ and vector $a = f(\nu) - \nabla f(\nu)\nu$.

To quantify the accuracy of a general N_Tth-order approximation, Taylor's Theorem [27, Sec. 9.6] can be rewritten as follows

$$f(x) = f_{N_T}(x) + R_{N_T}(x). \tag{6}$$

For a 1D function, we obtain remainder $R_{N_T}(x) := f(x) - f_{N_T}(x) = \sum_{l=N_T+1}^{\infty} \frac{f^{(l)}(\nu)}{l!}(x-\nu)^l$. Following [27, Sec. 9.6] we derive an alternative expression for the remainder for which an upperbound can be computed.

Proposition 1: *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is defined on a closed interval $\mathbb{G}$ with $\nu \in \mathbb{G}$ and $f^{(N_T+1)}$ exists on the same interval. Then for each $x \in \mathbb{G}$, there exists a ζ on the interval between ν and x, such that*

$$R_{N_T}(x) = \frac{f^{(N_T+1)}(\zeta)}{(N_T+1)!}(x-\nu)^{N_T+1}. \tag{7}$$

The proof of this proposition is based on the derivation in [27] and [32, Sec. 2.4]. Using Taylor's inequality [33], an upperbound of the remainder (7) can be found, which equals

$$|R_{N_T}(x)| \leq \sup_{\zeta \in \mathbb{G}} \left(\left| \frac{f^{N_T+1}(\zeta)}{(N_T+1)!} \right| \right) \cdot |x-\nu|^{N_T+1}, \tag{8}$$

and holds for all $x \in \mathbb{G}$. The derivation of this upperbound and its extension to multivariate functions is given in [32, Sec. 2.4]. By taking the supremum over $x \in \mathbb{G}$, an upperbound on the approximation error of the N_Tth-degree Taylor polynomial, denoted by remainder R_{N_T} can be computed.

Next we derive the additional approximation error from using a first-order Taylor polynomial as in (5) instead of an N_T-order Taylor polynomial. Denote this error by $R_1(x)$, which is equal to $R_1(x) = f_{N_T}(x) - f_1(x)$. For 1D functions, we use (3) and (4) to derive

$$R_1(x) = \sum_{l=2}^{N_T} \frac{f^{(l)}(\nu)}{l!}(x-\nu)^l. \tag{9}$$

Denote the bounded difference between $f(x_t)$ and its affine approximation $Ax_t + a$ for $x \in \mathbb{G}$ by $\kappa_t \in \mathcal{K} \subset \mathbb{R}^{n_x}$. In line with (6), we define approximation error $\kappa_t = f_{N_T}(x) - f_1(x) + R_{N_T}(x) = R_1(x) + R_{N_T}(x)$. Now, we can conclude the following.

Theorem 1: *Given a nonlinear function $f(x_t)$ that is sufficiently smooth, there exists a bounded vector $\kappa_t = R_1(x) + R_{N_T}(x)$, such that $f(x_t) = Ax_t + a + \kappa_t$ for $x \in \mathbb{G}$.*

The proof of this theorem follows from the extension of Proposition 1 to higher-dimensional functions as in [32].

Local finite-state abstraction. To synthesize a controller for a continuous-state system, we further approximate the behavior of its affine approximation by a finite-state abstract system. More precisely, we partition the state space $\mathbb{X}$ in a finite number of regions $\mathbb{A}_j \subset \mathbb{X}$, such that it covers the complete state space, that is $\bigcup_j \mathbb{A}_j = \mathbb{X}$ and there is no overlap between the regions $\mathbb{A}_j \cap \mathbb{A}_l = \emptyset$ for $j \neq l$. In each region, a representative point $\hat{x}_j \in \mathbb{A}_j$ is chosen. Together, these points make up the set of abstract states, $\hat{x} \in \hat{\mathbb{X}} = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{N_A}\}$. Besides that, a finite number of inputs is selected from $\mathbb{U}$ to form the abstract input space $\hat{\mathbb{U}}$. Consider the operator $\Pi : \mathbb{X} \to \hat{\mathbb{X}}$ that maps states from the original state space to the abstract state space. Then locally the dynamics of the abstract system equal

$$\hat{x}_{t+1} = \Pi(A\hat{x}_t + B\hat{u}_t + a + \hat{w}_t), \tag{10}$$

with states $\hat{x} \in \mathbb{G} \subset \hat{\mathbb{X}}$, initial state $\hat{x}_0 = \Pi(x_0)$, inputs $\hat{u} \in \hat{\mathbb{U}}$, and disturbances $\hat{w} \in \mathbb{W}$. The disturbance has realizations $\hat{w} \sim \mathbb{P}_{\hat{w}}$. Next, we introduce a bounded vector $\beta \in \mathcal{B} \subset \mathbb{R}^{n_x}$, that pushes the state to its representative point. Then, with a slight abuse of notation[2] the state dynamics of the local abstract system (10) for $\hat{x} \in \mathbb{G}$ satisfy

$$\hat{x}_{t+1} \in A\hat{x}_t + B\hat{u}_t + a + \hat{w}_t + \mathcal{B}.$$

More precisely, there exist $\beta \in \mathcal{B}$, such that $\hat{x}_{t+1} = A\hat{x}_t + B\hat{u}_t + a + \hat{w}_t + \beta_t$. Now, we can write the state dynamics of the local abstract system as an affine system whose behavior is described by

$$\hat{x}_{t+1} = A\hat{x}_t + B\hat{u}_t + a + \hat{w}_t + \beta_t \text{ for } \hat{x} \in \mathbb{G} \tag{11}$$

[2]Here, the Minkowski sum of the two sets is neglected.

Global finite-state abstraction. To define a global finite-state abstraction, we partition the state space $\mathbb{X}$ with polytopic cells $\hat{P}_i$ with $i \in \{1, \ldots, N_P\}$, such that $\bigcup_i \hat{P}_i = \mathbb{X}$ and such that the partitions do not overlap $\hat{P}_i \cap \hat{P}_j = \emptyset$ for $i \neq j$. For each of the cells, we can now compute a local finite-state approximation as described in the previous paragraph. That is, for each $\hat{P}_i$, we compute the abstract dynamics as (11) with $\hat{P}_i \subseteq \mathbb{G}$. As such, we can translate the abstract system (11) that locally approximates the nonlinear system (1) to a global piecewise-affine system that approximates the nonlinear system as

$$\hat{M}: \begin{cases} \hat{x}_{t+1} = A_i \hat{x}_t + B\hat{u}_t + a_i + \hat{w}_t + \beta_t \text{ for } \hat{x} \in \hat{P}_i \\ \hat{y} = C\hat{x}_t, \end{cases} \tag{12}$$

with states $\hat{x} \in \hat{\mathbb{X}} \subset \mathbb{X}$, initial state $\hat{x}_0$, inputs $\hat{u} \in \hat{\mathbb{U}}$ and disturbance $\hat{w} \in \mathbb{W}$.

IV. Piecewise stochastic simulation relation

In this section, we discuss how to quantify the difference between the original nonlinear stochastic model and the abstract, global finite-state model obtained via piecewise affine approximations.

A. Similarity quantification

We need to quantify the similarity between the models (1) and (12). We start by defining a local metric for the error dynamics based on coupling the models through its inputs and stochastic disturbances. First, we couple the inputs u and $\hat{u}$ by using an interface function denoted as

$$\mathcal{U}_v : \hat{\mathbb{U}} \times \hat{\mathbb{X}} \times \mathbb{X} \to \mathbb{U}. \tag{13}$$

This computes a control input u given the current input $\hat{u}$ of the abstract model and given the state $\hat{x}$ and x of the abstract and concrete models, respectively.

Next, we couple the disturbances w and $\hat{w}$ with respectively realizations $w \sim \mathbb{P}_w$ and $\hat{w} \sim \mathbb{P}_{\hat{w}}$ as in [28] using the following definition based on [34].

Definition 2 (Coupling probability measures): *A coupling of probability measures $\mathbb{P}_w$ and $\mathbb{P}_{\hat{w}}$ on the same measurable space $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$ is any probability measure $\mathcal{W}$ on the product measurable space $(\mathbb{W} \times \mathbb{W}, \mathcal{B}(\mathbb{W} \times \mathbb{W}))$ whose marginals are $\mathbb{P}_w$ and $\mathbb{P}_{\hat{w}}$, that is,*

$$\mathcal{W}(\hat{A} \times \mathbb{W}) = \mathbb{P}_{\hat{w}}(\hat{A}) \text{ for all } \hat{A} \in \mathcal{B}(\mathbb{W})$$
$$\mathcal{W}(\mathbb{W} \times A) = \mathbb{P}_w(A) \text{ for all } A \in \mathcal{B}(\mathbb{W}).$$

We can trivially extend this definition to Borel measurable stochastic coupling kernels

$$\mathcal{W} : \hat{\mathbb{U}} \times \hat{\mathbb{X}} \times \mathbb{X} \to \mathcal{P}(\mathbb{W}^2). \tag{14}$$

Now, we can introduce a simulation relation to quantify the similarity between the stochastic models M (1) and $\hat{M}$ (12) based on [11].

Definition 3: (ϵ,$\boldsymbol{\delta}$)-stochastic simulation relation): *Let stochastic models M and $\hat{M}$ with metric output space $(\mathbb{Y}, \boldsymbol{d}_{\mathbb{Y}})$, an interface function $\mathcal{U}_v$ (13), and a Borel measurable stochastic kernel $\mathcal{W}$ (14) be given. If there exists a measurable relation $\mathcal{R} \subseteq \hat{\mathbb{X}} \times \mathbb{X}$, with $(\hat{x}_0, x_0) \in \mathcal{R}$, and such that*

1) $\forall (\hat{x}, x) \in \mathcal{R} : \boldsymbol{d}_{\mathbb{Y}}(\hat{y}, y) \leq \epsilon$, *and*
2) $\forall (\hat{x}, x) \in \mathcal{R}, \forall \hat{u} \in \hat{\mathbb{U}} : (\hat{x}^+, x^+) \in \mathcal{R}$ *holds with probability at least* $1 - \boldsymbol{\delta}(\hat{x})$, *with* $\boldsymbol{\delta} : \hat{\mathbb{X}} \to [0, 1]$.

then $\hat{M}$ is $(\epsilon, \boldsymbol{\delta})$-stochastically simulated by M, and this simulation relation is denoted as $\hat{M} \preceq_\epsilon^{\boldsymbol{\delta}} M$.

We refer to ϵ as the (metric) output deviation and to $\boldsymbol{\delta}$ as the probabilistic or stochastic deviation function. Note that unlike [11] the stochastic deviation is not uniform for the whole state space. Instead it is introduced as a function $\boldsymbol{\delta} : \hat{\mathbb{X}} \to [0, 1]$ that depends on the abstract state $\hat{x}$. If $\boldsymbol{\delta}(\hat{x})$ is a piecewise constant function, then we refer to the simulation relation as a *piecewise stochastic simulation relation*. We have defined a measure to quantify the difference between two models on a global level, that is, over the full state space. The question is now how we can compute it based on the given local piecewise-affine structure of the abstractions.

B. Piecewise similarity quantification

Consider a simulation relation given as

$$\mathcal{R} := \left\{ (\hat{x}, x) \in \hat{\mathbb{X}} \times \mathbb{X} \mid ||x - \hat{x}||_D \leq \epsilon \right\}, \tag{15}$$

with a weighting matrix D such that the first condition of Def. 3 is satisfied, which reduces to the requirement that

$$C^\top C \preceq D. \tag{16}$$

In this subsection, we will use this relation (15) to show that a global (ϵ,$\boldsymbol{\delta}$)-stochastic simulation relation can be efficiently computed with a piecewise constant probability deviation function $\boldsymbol{\delta} : \hat{\mathbb{X}} \to [0, 1]$ defined based on the state partitioning $\hat{P}_i$

$$\boldsymbol{\delta}(\hat{x}) = \delta_i \text{ if } \hat{x} \in \hat{P}_i.$$

The function $\boldsymbol{\delta}$ assigns a constant local probability deviation to each partition in the abstract state space $\hat{\mathbb{X}}$ based on a local similarity quantification that is derived using the local stochastic error dynamics.

Local stochastic error dynamics. Consider a local interface function $u_t = \mathcal{U}_{vi}(\hat{u}_t, \hat{x}_t, x_t)$ as

$$u_t = \hat{u}_t + K_{f,i}(x_t - \hat{x}_t), \tag{17}$$

with feedback matrix $K_{f,i} \in \mathbb{R}^{n_u \times n_x}$ and a local stochastic kernel $\mathcal{W}_i$, assigning to each $(u, \hat{x}, x)$ a probability measure

$$\mathcal{W}_i : \hat{\mathbb{U}} \times \hat{P}_i \times \mathbb{X} \to \mathcal{P}(\mathbb{W}^2). \tag{18}$$

We have that

$$\hat{x}_{t+1} = A_i \hat{x}_t + B\hat{u}_t + a_i + \hat{w}_t + \beta_t \text{ for } \hat{x} \in \hat{P}_i$$

furthermore if $\|x - \hat{x}\|_D \leq \epsilon$ then there exists a κ_t such that

$$x_{t+1} = A_i x_t + Bu_t + a_i + w_t + \kappa_t \text{ with } \kappa_t \in \mathcal{K}_i.$$

Given that x_t belongs to P_i defined as

$$P_i := \{x \in \mathbb{X} | \exists \hat{x} \in \hat{P}_i : \|x - \hat{x}\|_D \leq \epsilon\}.$$

Following the previous section, the set $\mathcal{K}_i$ is defined as

$$\mathcal{K}_i := \sup_{x \in P_i} \left(R_1(x) + R_{N_T}(x)\right),$$

with $R_1(x)$ as in (9) and R_{N_T} as in (7) with upperbound (8).

If the states satisfy $\hat{x}_t \in \hat{P}_i$ and $x_t \in P_i$, then the error dynamics of $x_{\Delta t} := x_t - \hat{x}_t$ equal

$$x_{\Delta t+1} = (A_i + BK_{f,i})x_{\Delta t} + (w_t - \hat{w}_t) + \kappa_t - \beta_t \quad (19)$$

with $\kappa \in \mathcal{K}_i$ and $\beta \in \mathcal{B}$ and with $(\hat{w}_t, w_t) \sim \mathcal{W}_i$.

Local coupling and interface functions with $\boldsymbol{\delta} = \delta_i$. Following [28], we make sure that the second condition of Def. 3 is satisfied by finding a global invariant set $\{x_\Delta \,|\, \|x_\Delta\|_D \leq \epsilon\}$ parameterized with a global D for the error dynamics (19). Together with D, we have to compute an optimal local interface function (17) and coupling (18) for all partitions $\hat{P}_i$, with $i \in \{1, \dots N_p\}$. More precisely, we design $\mathcal{W}_i$ and $K_{f,i}$ such that the probability $1 - \delta_i$ with which $\|x_{\Delta t+1}\|_D \leq \epsilon$ holds is maximized. To this end, we consider a local coupling for which $\hat{w} = w + F_i(x - \hat{x})$ holds with probability $1 - \delta_i$. The use of the coupling term F_i introduced in [28] reduces the complexity of the design of $\mathcal{W}_i$ as it allows us to write the design problem as a set of implications or parameterized matrix inequalities. That is, a relation between this term and the probability deviation δ_i can been derived as upperbound $||F_i(x - \hat{x})|| \leq r_i$ with

$$r_i := \left| 2 \operatorname{idf}\left(\frac{1 - \delta_i}{2}\right)\right|. \quad (20)$$

Here, idf denotes the inverse distribution function of a Gaussian distribution $\mathcal{N}(0, I)$.

As can be concluded from the error dynamics in (19), together with the coupling, the interface function can be used to further compensate for the error in the state by computing a suitable feedback-term $K_{f,i}$. To make sure that the bound $u \in \mathbb{U}$ is satisfied, we reduce $\hat{\mathbb{U}}$ accordingly and add an upperbound on the feedback-term as $||K_{f,i}(x - \hat{x})|| \leq u_u$.

Taken together, we can conclude the following.

Lemma 1 (Piecewise requirements): *Consider stochastic models M (1) and $\hat{M}$ (12) for which a simulation relation $\mathcal{R}$ (15) with weighting matrix D satisfying (16) is given. If there exist matrices F_i, and $K_{f,i}$ such that the following implications are satisfied for a given $\boldsymbol{\delta}(\hat{x})$*

$$x_\Delta^\top D x_\Delta \leq \epsilon^2 \implies \begin{cases} x_\Delta^\top F_i^\top F_i x_\Delta & \leq r_i^2 \\ x_\Delta^\top K_{f,i}^\top K_{f,i} x_\Delta & \leq u_u^2 \\ x_{\Delta t+1}^\top D x_{\Delta t+1} & \leq \epsilon^2, \end{cases} \quad (21)$$

with $x_{\Delta t+1}$ in (19) and r_i in (20), then there exists coupling kernels $\mathcal{W}_i$ and interfaces $\mathcal{U}_{vi}$ such that

$$\forall (\hat{x}, x) \in \hat{P}_i \times \mathbb{X}, \forall \hat{u} \in \hat{\mathbb{U}} : (\hat{x}^+, x^+) \in \mathcal{R} \quad (22)$$

holds with probability $1 - \delta_i$ for all $\hat{P}_i$, with $i \in \{1, \dots N_p\}$ and with $\bigcup_i \hat{P}_i = \hat{\mathbb{X}}$.

The first and second implication in (21) are sufficient conditions for the bounds on respectively the *coupling compensator term* and the feedback-term $K_{f,i}(x - \hat{x})$. The last implication is a sufficient condition for a local version of the second condition of a $(\epsilon, \boldsymbol{\delta})$-stochastic simulation relation as in Def. 3. The full proof of this Lemma is omitted, since it follows the proof of Theorem 9 in [28]. Furthermore, in [28], it is shown how to write such implications (21) as parameterized matrix inequalities for efficient computation.

From local to global similarity quantification. To obtain a global similarity quantification, we define a global piecewise stochastic kernel $\mathcal{W}$ and a global interface function. Since $\hat{\mathbb{U}} \times \hat{P}_i \times \mathbb{X}$ for $i = 1, \dots, N_P$ is a partitioning of $\hat{\mathbb{U}} \times \hat{\mathbb{X}} \times \mathbb{X}$ we can use the local stochastic coupling kernel $\mathcal{W}_i : \hat{\mathbb{U}} \times \hat{P}_i \times \mathbb{X} \to \mathcal{P}(\mathbb{W}^2)$ for $i = 1, \dots, N_P$ to compute the global stochastic coupling kernel $\mathcal{W} : \hat{\mathbb{U}} \times \hat{\mathbb{X}} \times \mathbb{X} \to \mathcal{P}(\mathbb{W}^2)$ as

$$\mathcal{W}(\cdot \mid \hat{u}, \hat{x}, x) = \mathcal{W}_i(\cdot \mid \hat{u}, \hat{x}, x) \text{ if } \hat{x} \in \hat{P}_i. \quad (23)$$

Furthermore, the interface function can similarly be composed as

$$\mathcal{U}_v(\hat{u}_t, \hat{x}_t, x_t) = \mathcal{U}_{vi}(\hat{u}_t, \hat{x}_t, x_t) \text{ if } \hat{x} \in \hat{P}_i. \quad (24)$$

We can now show that these functions constitute to a $(\epsilon, \boldsymbol{\delta})$-stochastic simulation relation for the given simulation relation.

Theorem 2 (Piecewise stochastic similarity): *Let stochastic models M (1) and $\hat{M}$ (12) be given. Then the interface function $\mathcal{U}_v$ (24) and the global Borel measurable stochastic kernel $\mathcal{W}$ (23) computed for the simulation relation (15) based on (21) define an (ϵ, $\boldsymbol{\delta}$)-stochastic simulation relation in a piecewise manner as given in Def. 3 if*

- *it holds that $(\hat{x}_0, x_0) \in \mathcal{R}$, and if*
- *the simulation relation satisfies matrix inequality (16).*

Proof. The proof builds on Lemma 1, and can be sketched as follows. The first condition of Def. 3 holds by choosing matrix D, such that (16) holds. This is proven in the proof of Theorem 9 in [28]. Lemma 1 shows that if (21) is satisfied then a local stochastic kernel $\mathcal{W}_i$ as in (18) exists, such that (22) holds with probability $1 - \delta_i$. By choosing the interface function $\mathcal{U}_v$ as (24), and by choosing the global stochastic kernel as in (23) the second condition in Def. 3 is satisfied. □

V. Temporal logic control

In this section, we discuss how to compute the satisfaction probability of temporal logic specifications based on the dynamic programming mappings from [29]. Next, we show how to apply the method from this paper to design a provably correct controller for a nonlinear stochastic case study and discuss the results.

A. Dynamic programming

In correct-by-design control synthesis, an scLTL specification ϕ can be written as a *deterministic finite-state automaton* (DFA), that is characterized by the tuple $\mathcal{A}_\phi = \{Q, q_0, \Sigma, \tau_\mathcal{A}, F\}$ [9]. Here, the set of states is denoted by Q and initialized by the initial state q_0. The input alphabet and transition function are respectively denoted by $\Sigma = 2^{AP}$ and $\tau_\mathcal{A} : Q \times \Sigma \to Q$. Finally, F denotes the set of accepting states. The word $\boldsymbol{\pi}$ satisfies the specification ϕ, that is $\boldsymbol{\pi} \models \phi$, as long as the the word $\boldsymbol{\pi}$ is *accepted* by the DFA $\mathcal{A}_\phi$. This means that there exists a trajectory $q_0, q_1, \ldots, q_f$ with $q_f \in F$ starting at q_0 and evolving according to $q_{t+1} = \tau_\mathcal{A}(q_t, \pi_t)$. By analyzing the product composition between the system M and the specification DFA $\mathcal{A}_\phi$, denoted as $M \otimes \mathcal{A}_\phi$, we can compute the satisfaction probability. The composition $M \otimes \mathcal{A}_\phi$ consists of states $(x_t, q_t) \in \mathbb{X} \times Q$. For a given input u_t, it evolves from (x_t, q_t) to (x_{t+1}, q_{t+1}) by following the stochastic transition from x_t to x_{t+1} in (1) and from q_t to $q_{t+1} = \tau_\mathcal{A}(q_t, L(Cx_t))$. Therefore, computing the satisfaction probability is equivalent to solving a reachability problem of the composition $M \otimes \mathcal{A}_\phi$, which can be written as a dynamic program. With a slight abuse of notation we will also refer to the stationary policy of this composed system as $\boldsymbol{\mu} : \hat{\mathbb{X}} \times Q \to \hat{\mathbb{U}}$.

We use the abstract model to compute the satisfaction probability, since this is not possible for the original model due to its continuous states. The satisfaction probability with policy $\boldsymbol{\mu}$ in the time horizon $[1, \ldots, N]$ is expressed using the value function $V_N^{\boldsymbol{\mu}}(\hat{x}, q) : \hat{\mathbb{X}} \times Q \to [0, 1]$. This is equivalent to the probability that the trajectory generated by applying $\boldsymbol{\mu}$ to $M \otimes \mathcal{A}_\phi$ and starting at (x, q) reaches the target set F within this time horizon. The value function $V_N^{\boldsymbol{\mu}}(\hat{x}, q)$ is defined as

$$V_N^{\boldsymbol{\mu}}(\hat{x}, q) := \mathbb{E}_{\boldsymbol{\mu}} \left(\max_{0 \le t \le N} \mathbf{1}_F(q_t) | (\hat{x}_0, q_0) \right),$$

with indicator function $\mathbf{1}_F$ that is equal to 1 if $q \in F$ and 0 otherwise. The value function can also be computed recursively for a policy $\boldsymbol{\mu}_i = (\mu_{i+1}, \ldots, \mu_N)$ with horizon $N - i$ as follows

$$V_{N-k+1}^{\boldsymbol{\mu}_{k-1}}(\hat{x}, q) = \mathbf{T}^{\mu_k}(V_{N-k}^{\boldsymbol{\mu}_k})(\hat{x}, q)$$

initialized with $V_0 \equiv 0$. Here, the operator $\mathbf{T}^{\mu_k}(\cdot)$ is defined as follows

$$\mathbf{T}^{\mu_k}(V)(\hat{x}, q) := \mathbb{E}_{\mu_k}(\max\left\{1_F(q^+), V(\hat{x}^+, q^+)\right\}),$$

with input DFA transitions $q^+ = \tau_{\mathcal{A}_\phi}(q, L(Cx^+))$. For a stationary policy $\boldsymbol{\mu}$, the infinite-horizon value function is computed as $V_\infty^{\boldsymbol{\mu}} = \lim_{N \to \infty} (\mathbf{T}^{\boldsymbol{\mu}})^N V_0$ initialized with $V_0 \equiv 0$. The policy-optimal converged value function V_∞^* is computed with the operator $\mathbf{T}^*(\cdot) := \sup_\mu \mathbf{T}^\mu(\cdot)$. The corresponding satisfaction probability can now be computed as $\mathbb{P}^{\boldsymbol{\mu}} := \max(\mathbf{1}_F(\bar{q}_0, V_\infty^*(x_0, \bar{q}_0)))$ with $\bar{q}_0 = \tau(q_0, L(Cx_0))$.

To cope with the output deviation ϵ and with probability deviations described by the function $\boldsymbol{\delta}(\hat{x})$, we define a robust dynamic programming mapping similar to [11], as

$$\mathbf{T}_{\epsilon,\boldsymbol{\delta}}^{\mu_k}(V)(\hat{x}, q) := \mathbf{L}\left(\mathbb{E}_\mu(\min_{q^+ \in Q^+} \max\left\{1_F(q^+), V(\hat{x}^+, q^+)\right\}) - \boldsymbol{\delta}(\hat{x}) \right),$$

with $\mathbf{L} : \mathbb{R} \to [0, 1]$ a truncation function defined as $\mathbf{L}(\cdot) := \min(1, \max(0, \cdot))$ and with

$$Q^+(q, \hat{y}^+) := \left\{ \tau_\mathcal{A}(q, L(y^+)) \mid ||y^+ - \hat{y}^+|| \le \epsilon \right\}.$$

We can now compute the robust satisfaction probability by considering the first time instance based on x_0, that is,

$$\mathbb{R}^{\boldsymbol{\mu}} := \max(\mathbf{1}_F(\bar{q}_0, V_\infty^{\boldsymbol{\mu}}(x_0, \bar{q}_0)))$$

with $\bar{q}_0 = \tau_\mathcal{A}(q_0, L(Cx_0))$. This probability is robust in the sense that it gives a lower-bound on the probability stated in (2), i.e., $\mathbb{P}(M \times \boldsymbol{C} \models \phi) \ge \mathbb{R}_{\epsilon,\boldsymbol{\delta}}(\hat{M} \times \hat{\boldsymbol{C}} \models \phi)$.

B. Case study

We have applied this method to a forced, stochastically perturbed Van der Pol oscillator with state $x_t = [x_{1,t}, x_{2,t}]^\top$ and state dynamics[3]

$$x_{1t+1} = x_{1t} + x_{2t}\tau + w_{1t}$$
$$x_{2t+1} = f_2(x_t) + u_t + w_{2t},$$

with nonlinear function

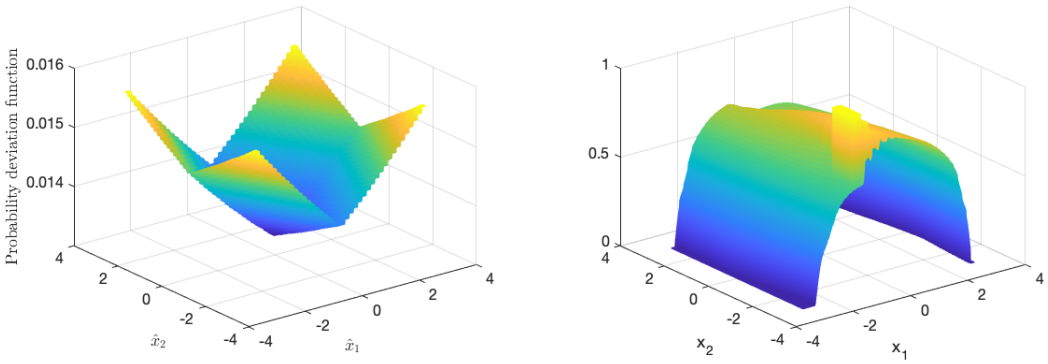
$$f_2(x_t) = x_{2t} + (-x_{1t} + (1 - x_{1t}^2)x_{2t})\tau).$$

Here, $\tau = 0.1$ is the sampling time and $w \sim \mathcal{N}(0, 0.2I_2)$ is a Gaussian disturbance. Furthermore, we have considered states $x \in \mathbb{X} = [-3, 3]^2$, input $u \in [-1, 1]$, a safe region $P_1 = \mathbb{X}$ and a goal region $P_2 = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid -1.2 \le x_1 \le -0.9, -2.9 \le x_2 \le -2\}$. The goal of the controller is to guarantee that the system reaches the goal region, while staying in the safe region. This can be written using scLTL as $\phi = P_1 \cup P_2$.

We obtained an abstract model with state dynamics as in (10) by partitioning the state space with square regions of width 0.01 leading to[4] $\beta \in \mathcal{B} = [-0.01, 0.01]^2$ and with $\hat{u} \in \hat{\mathbb{U}} = [-0.6, -0.3, 0, 0.3, 0.6]$ leaving some input action for the feedback part, namely $-0.4 \le K_{f,i}(x - \hat{x}) \le 0.4$. Next, we performed a piecewise-affine approximation by using 1600 equally sized square partitions to obtain a piecewise-affine abstraction as in (12). We subsequently selected $\epsilon = 0.08$ and computed a corresponding probability deviation function $\boldsymbol{\delta}(\hat{x})$ such that the implications in (21) are satisfied. We computed the global stochastic kernel $\mathcal{W}$ based on (23) and interface function $\mathcal{U}_v$ as in (24) and used Theorem 2 to obtain an $(\epsilon, \boldsymbol{\delta})$-stochastic simulation relation. Finally, we used the dynamic programming mappings as discussed in Section V-A to compute a robust controller $\boldsymbol{C}$, while simultaneously computing the robust satisfaction

[3]Here, the first suffix of x_{1t} refers to the dimension, while the second suffix $_t$ indicates the time.

[4]Normally, you would get $\beta \in \mathcal{B} = [-0.005, 0.005]^2$, however, our `Matlab` implementation uses an efficient tensor-based computation that leads to a bigger set for β.



(a) Probability deviation function $\boldsymbol{\delta}(\hat{x})$

(b) Satisfaction probability $\mathbb{R}_{\epsilon,\boldsymbol{\delta}}(\hat{M} \times \hat{\boldsymbol{C}} \models \phi)$

Fig. 1: Probability deviation function for the abstract state space (left) and robust satisfaction probability (right) when starting at initial state $x_0 = [x_1(0), x_2(0)]^\top$.

probability. The probability deviation function $\boldsymbol{\delta}(\hat{x})$ and the satisfaction probability $\mathbb{R}_{\epsilon,\boldsymbol{\delta}}(\hat{M} \times \hat{\boldsymbol{C}} \models \phi)$ are given in Fig. 1 and are computed in `Matlab` in approximately 82 minutes, while using approximately 64Mb memory. More specifically, 60% of the computation time is spent on gridding the state space and 35% on computing the parametrized matrix inequalities such that the implications in Lemma 1 are satisfied. Furthermore, the memory usage is computed based on the sizes of the variables stored in the workspace.

Comparison to available software tools. Similar case studies have been presented in [13], [35], where [13] considers an autonomous Van der Pol oscillator and [35] combines the input with a multiplicative noise term. However, the results presented in [13], [35] are limited to verification or a reachability analysis instead of the control synthesis performed in this paper.

Furthermore, we have chosen a more stochastic variant of the case study with a Gaussian disturbance with variance $0.2I_2$ instead of an uniform distribution with support $[-0.02,\ 0.02] \times [-0.02,\ 0.02]$ as used in [13], [35]. The unbounded nature of the Gaussian disturbance contributes significantly to the difficulty of this case study.

VI. Conclusions

Concluding, to the best of our knowledge this paper is the first to describe a control synthesis method for general temporal logic control that uses piecewise-affine approximations of stochastic nonlinear models. By using a state-dependent probability deviation, a lower-bound on the satisfaction probability is computed as shown in the case study. For future work, more advanced piecewise-affine approximations, such as [24]–[26] can be applied and a more detailed analysis of the results will be investigated. Besides that, the computation time of solving the parameterized linear matrix inequalities in our `Matlab` implementation should be decreased as well.

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